Trust Hole Identification in Signed Networks

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Abstract. In the trust-centric context of signed networks, the social links among users are associated with specific polarities to denote the attitudes (trust vs distrust) among the users. Different from traditional unsigned social networks, the diffusion of information in signed networks can be affected by the link polarities and users' positions significantly. In this paper, a new concept called "*trust hole*" is introduced to characterize the advantages of specific users positions in signed networks. To uncover the trust holes, a novel trust hole detection framework named "Social Community based tRust hOLe expLoration" (SCROLL) is proposed in this paper. Framework SCROLL is based on the signed community detection technique. By removing the potential trust hole candidates, SCROLL aims at maximizing the community detection cost drop to identify the optimal set of trust holes. Extensive experiments have been done on real-world signed network datasets to show the effectiveness of SCROLL.

Keywords: Trust Hole Detection; Signed Networks; Data Mining

1 Introduction

In traditional works on sociology and social networks, the concept *structural hole* refers to individuals who act as intermediaries or bridges between others who are not directly connected [9]. Via these *structural holes*, information can propagate to separated individuals in different communities, or those who are otherwise not interacting with each other. As a result, the *structural holes* who take these bridging positions in society or social networks will accrue significant advantages than other users [9]. In traditional social networks with regular friendship connections among users, structural holes related problems have been studied for years, and dozens of papers on it have already been published [1, 3, 5, 9].

Meanwhile, in some online social networks like Epinions[1], the connections connected to users are associated with specific polarities (e.g., *positive* vs *negative*) to denote different attitudes among users (e.g., *trust* vs *distrust*). Such a kind of online social networks are formally represented as the *signed networks* [11]. Different from traditional regular unsigned social networks, in the trust-centric context of signed networks,

[1] http://www.epinions.com

diffusion of information can be affected by the link polarities significantly. For instance, in signed networks, information tends to propagate via the trust links between users who trust each other instead of those distrusted ones. Viewed in this way, users who bridge different cliques via distrust links actually cannot transmit information across these cliques. Therefore, the traditional *structural holes* (i.e., the inter-community users in unsigned networks) concept [9] can no longer work for the signed networks.

To characterize the advantages of specific users' positions in signed networks, a new concept named *trust holes* is introduced in this paper. Depending on the polarities of links attached to them, the *trust hole* concept has two variants: (1) *positive trust holes* who connect multiple isolated social communities via positive links, and (2) *negative trust holes* who connect users within communities via negative links instead. Via the positive trust holes, information can propagate between different social communities, as people will trust information propagated from these hole users. Meanwhile, via the negative trust holes, the intra-community information dissemination will be blocked instead, as few of the neighbors will believe the information from the people they distrust. Therefore, the positions of both *positive trust holes* and *negative trust holes* will have great advantages in passing information among users in signed networks. The formal definition of the *trust hole* concept is available in Section 2. Specifically, the inter-community nodes attached with negative links and the intra-community nodes attached with positive links are not *trust holes*, as their position own no advantages in propagating information in signed networks. We will clarify that in detail in Section 2.

Problem Studied: In this paper, we aim at identifying the *trust holes* from the signed networks, and the problem is referred to as the "Signed network trust HolE iDentification" (SHED) problem formally.

The SHED problem is an interesting research problem, and it is also very important for many concrete applications, e.g., community structure [6,10], and information diffusion [13,14] (existence of the holes can help disseminate the information more broadly) in signed networks. In addition, the SHED problem is a novel problem and we are the first to study it in signed networks. Different from existing works about structural holes in unsigned networks [3,9], the networks studied in this paper are *signed networks*, and the target to be identified are the *trust holes* instead. For more information about related works, please refer to Section 5.

The SHED problem is very challenging to solve due to the following reasons:

- *definition of trust hole*: The *trust hole* proposed in this paper is a new concept. A formal definition of the *trust hole* concept is needed before studying the SHED problem.
- *formulation of the* SHED *problem*: In the signed network setting, how to formulate the SHED problem with clear motivations and objectives is still an open problem.
- *solution to the* SHED *problem*: The SHED itself is a difficult problem. Some trivial methods, like isolated trust hole identification in the positive sub-graph (and negative sub-graph) along, will face great challenges in both obtaining the positive and negative trust holes independently and fusing the trust hole results from these two sub-graphs to get the final consistent results. An integrated *trust hole* detection framework based on the whole signed network is desired.

To address the above challenges, an integrated trust hole detection framework named SCROLL (Social Community based tRust hOLe expLoration) is proposed in this paper. Before introducing SCROLL, we will first define the *trust hole* concept with full considerations about the link polarities in the signed networks. SCROLL formulates the SHED problem from the community detection perspective. By removing the potential trust hole users, SCROLL aims at maximizing the community detection cost drop to identify the optimal set of trust hole candidates, which maps the SHED to a max-min optimization problem. A new concept named "*signed normalized cut decrease*" is proposed in SCROLL to quantify the cost drop formally based on the *signed normalized cut* measure introduced in this paper. The SHED problem is shown to be NP-hard, but based on such a formulation, SCROLL can solve the SHED approximately with an alternative updating schema based schema.

The following parts of this paper are organized as follows. Terminology definition and problem formulation are given in Section 2. The method is introduced in Section 3, which is evaluated in Section 4. Finally, Section 5 is about the related works and Section 6 concludes this paper.

2 Problem Formulation

2.1 Terminology Definition

The networks studied in this paper are *signed networks*, where links are associated with different polarities.

Definition 1 (Signed Network): A signed network can be represented as $G = (\mathcal{V}, \mathcal{E}, s)$, where $\mathcal{V}$ ($|\mathcal{V}| = n$) and $\mathcal{E}$ ($|\mathcal{E}| = m$) are the sets of users and links respectively. Sign mapping $s : \mathcal{E} \rightarrow \{+1, -1\}$ projects links to their different polarities, where polarities $+1$ and -1 denote that the links are the trust and distrust links respectively.

Users in regular social networks will form social communities based on the connections among them, where intra-community connections are more dense compared with those between different communities [17]. The *social communities* formed in signed networks can be different due to the polarities attached to links.

Definition 2 (Signed Social Community): Given a signed network G, we can represent the communities formed by users in G as $\mathcal{C} = \{C_1, C_2, \cdots, C_k\}$, where k is the community number, $C_i \subseteq \mathcal{V}, \forall i \in \{1, 2, \cdots, k\}$ and $\bigcup_{i=1}^{k} C_i = \mathcal{V}$. Generally speaking, in the trust-centric context, users connected by positive links tend to trust each other and will be grouped in the same community. Meanwhile, for those connected by distrust links, they will have very few social interactions and will be partitioned into different communities.

However, in the real scenario, the signed social communities formed by people cannot fit the definitions exactly, and there may still exist a large number of inter-community positive links and intra-community negative links. In such a case, the positions of individuals connecting different communities via positive links, as well as

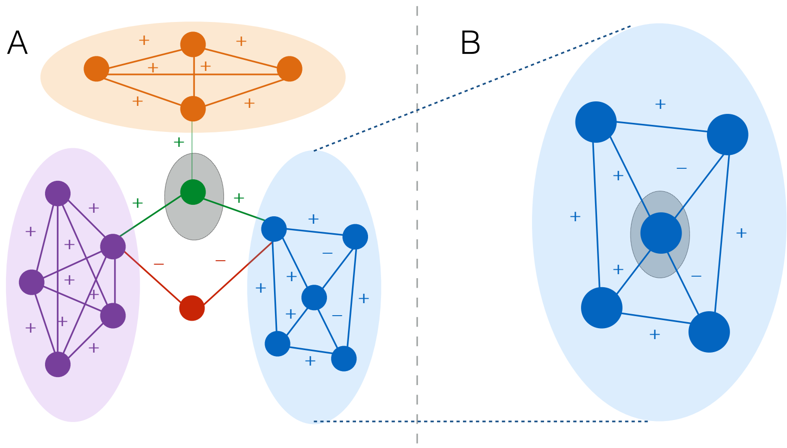


Fig. 1. Trust Holes vs Structural Hole. (A: positive trust hole, B: negative trust hole)

those connecting individuals within the same community via negative links will have significant advantages in information dissemination (as introduced in Section 1).

Definition 3 (Signed Trust Hole): Given a signed network G (with signed social community $\mathcal{C}$), literally, the *signed trust holes* in G denote a subset of users in G (i.e., $\mathcal{H} \subset \mathcal{V}$) occupying positions of the largest advantages. More specifically, the signed trust holes in $\mathcal{H}$ are the users either (1) connecting different communities via positive links (connected with users in them), who are referred to as the *positive trust holes*; or (2) connecting users within the same community with negative links, who are called the *negative trust holes* respectively.

To help illustrate this concept more clearly, we also give an example in Figure 1, where the colored regions denote different communities in the network. In plots A and B, we show the signed networks, where the links are associated with different polarities (i.e., positive vs negative). In the signed networks, nodes bridging different groups are not necessarily the trust holes. For instance, in plot A, the "*Green*" nodes which connecting different groups via positive links is defined as the positive trust hole, while the "*Red*" node bridging groups with negative links is not, as information will not propagate between groups via him/her. Besides the inter-community nodes, in signed networks, the intra-community nodes can also be the *trust holes*. For instance, in plot B, we observe that, in the "*Blue*" group, the central node connects to other nodes via both positive and negative links, which will partially block the dissemination of information within the group as some of his neighbors distrust information from him/her (he/she is still in the group as some others tend to trust him). The remaining intra-community nodes are not *trust holes* on the other hand. As a result, the positions of both the "*Green*" node in plot A and the central "*Blue*" node in plot B have significant advantages, which are called the *positive* and *negative trust holes* respectively.

2.2 Problem Statement

In this paper, we aim at identifying the set of *trust holes* from signed networks. Let $G = (\mathcal{V}, \mathcal{E}, s)$ be the signed network and $\mathcal{C}$ be the community structures detected from G. Various cost functions can be utilized to measure the quality of the detected community structure $\mathcal{C}$, which can be denoted as cost$(\mathcal{C}, G)$. Meanwhile, let $G' = (\mathcal{V}', \mathcal{E}')$ be the network obtained after removing the *positive* and *negative trust holes*, where $\mathcal{V}' = \mathcal{V} \backslash \mathcal{H}$ and $\mathcal{E}' = \{(u, v)|(u, v) \in \mathcal{E}, u \notin \mathcal{H}, v \notin \mathcal{H}\}$, and $\mathcal{C}'$ be the new community structures of G', which will lead to cost cost$(\mathcal{C}', G')$.

According to the definition of *trust holes*, the existence of *positive/negative trust hole* will not only influence the dissemination of information, but also blurring the network community structure. Removal of the *trust holes* from network G will also delete the inter-community positive links and intra-community negative links attached to them, and better community structures can be identified from G.

Therefore, in this paper, we propose to formulate the SHED problem from the community detection perspective. The optimal *trust holes* set $\mathcal{H}$ of size h can be identified by removing potential trust hole candidates from the network. The users removal of whom introducing the maximum community detection cost drop will be the optimal result. Formally, the objective function of the SHED problem can be represented as

$$\begin{aligned} &\max_{\mathcal{H}} \text{Cost}(\mathcal{C}^*, G) - \text{Cost}((\mathcal{C}')^*, G') \\ &s.t. \quad |\mathcal{H}| = h, \end{aligned}$$

where Cost$(\cdot)$ denote the costs introduced by the community structure in the network, and its concrete representations will be introduced in the following sections. And $\mathcal{C}^* = \arg\min_{\mathcal{C}} \text{Cost}(\mathcal{C}, G)$ and $(\mathcal{C}')^* = \arg\min_{\mathcal{C}'} \text{Cost}((\mathcal{C}'), G')$ denote the optimal community structure introducing the minimum costs in networks G and G' respectively.

Meanwhile, identification of the trust hole number (i.e., h) can be another interesting problem, but it is out of the scope of this paper, and we will leave it as a future work. In the SHED problem, the trust hole number is pre-given but we will also analysis the the effects of different parameter h on the performance of different comparison methods in the experiment section.

3 Proposed Methods

In this section, we will introduce the method SCROLL in detail. Based on the community cost function introduced in Section 3.1, we will provide the objective function of the SHED problem in Section 3.2 based on the "*signed normalized cut decrease*" concept. With some simple analysis, the SHED problem is shown to be NP-hard. An alternative updating schema based solution will be applied to address the objective function in Section 3.3.

3.1 Signed Normalized Cut Cost Function

Any community quality measures, e.g., *entropy*, *normalized dbi*, can be applied to define the community cost function. In this paper, we propose to use the *normalized* *cut* [20]. In this part, we will first do a quick review about the *normalized cut* measure for unsigned networks. Next, we propose to extend it to the *signed network* setting with full considerations of the constraints introduced by link polarities.

Traditional Normalized Cut Measure for Unsigned Networks Given a traditional unsigned social network G^u, based on the connection among users in which, we can define its adjacency matrix as $\mathbf{A}^u$. Its corresponding Laplace matrix can be represented as $\mathbf{L}^u = \text{Diag}(\mathbf{A}^u) - \mathbf{A}^u$, where $\text{Diag}(\mathbf{A}^u)$ denotes the corresponding diagonal matrix of $\mathbf{A}$ and $\text{Diag}(A^u)(i,i) = \sum_j A(i,j)$. Meanwhile, given the social structure $\mathcal{C}^u = \{C_1^u, C_2^u, \cdots, C_k^u\}$, we can define the corresponding indicator matrix as $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k)$, where $\mathbf{x}_j = (x_{1,j}, x_{2,j}, \cdots, x_{n,j})^\top$, and entry $\mathbf{x}_j(i) = x_{i,j}$ denotes whether user u_i is in cluster C_j^u or not. Traditional unsigned *normalized cut* cost function [16] is defined to be

$$\text{Ncut}(\mathcal{C}^u, G^u) = \sum_{i=1}^{i=k} \mathbf{x}_i^\top \mathbf{L}^u \mathbf{x}_i = \text{Tr}(\mathbf{X}^\top \mathbf{L}^u \mathbf{X}),$$

where $\text{Tr}(\cdot)$ denotes the trace of a matrix, and $\mathbf{X}$ is subject to constraint $\mathbf{X}^\top \mathbf{X} = \mathbf{I}$.

Extended Signed Normalized Cut Measure for Signed Networks However, in signed networks studied in this paper, the polarities associated to links can post extra constraints [22] on the community structures:

- *constraint of positive links*: From the trust-centric point of view, trust links (i.e., positive links) are stronger indicators of the closeness among users in signed networks. Generally, users who trust each other are more likely to share information and can be in the same community.
- *constraint of negative links*: Meanwhile, on the other hand, distrust links (i.e., negative links) can show the negative attitudes among users in signed networks. Users who distrust each other tend to have less social interactions, and will stay in different communities.

To handle the constraints introduced by the polarities of these signed links, in this paper, we propose to extend the traditional *normalized cut* concept to the trust-centric signed networks. Based on the positive links in network signed network G, we propose to construct the positive Laplace matrix $\mathbf{L}^+$. The cost introduced by detected communities $\mathcal{C}$ in cutting positive links can be represented as the *positive normalized cut* cost function:

$$\text{Ncut}(\mathcal{C}, G)^+ = \sum_{i=1}^{i=k} \mathbf{x}_i^\top \mathbf{L}^+ \mathbf{x}_i = \text{Tr}(\mathbf{X}^\top \mathbf{L}^+ \mathbf{X}).$$

Meanwhile based on the negative links in signed network G, we can construct the negative Laplace matrix $\mathbf{L}^-$. The cost introduced by detected communities $\mathcal{C}$ in cutting the negative links can be represented as the following *negative normalized cut* cost function:

$$\text{Ncut}(\mathcal{C}, G)^- = \sum_{i=1}^{i=k} \mathbf{x}_i^\top \mathbf{L}^- \mathbf{x}_i = \text{Tr}(\mathbf{X}^\top \mathbf{L}^- \mathbf{X}).$$

By considering the polarities of links in signed networks, in the trust-centric context, the optimal community structure should cut the minimum positive links but the maximum negative links. Viewed in this way, we introduce the *signed normalized cut* cost function for network G to be

$$\begin{aligned}
\text{Ncut}(\mathcal{C}, G) &= \alpha \cdot \text{Ncut}(\mathcal{C}, G)^+ - (1-\alpha) \cdot \text{Ncut}(\mathcal{C}, G)^- \\
&= \alpha \cdot \text{Tr}(\mathbf{X}^\top \mathbf{L}^+ \mathbf{X}) - (1-\alpha) \cdot \text{Tr}(\mathbf{X}^\top \mathbf{L}^- \mathbf{X}) \\
&= \text{Tr}\left(\mathbf{X}^\top (\alpha \cdot \mathbf{L}^+ - (1-\alpha) \cdot \mathbf{L}^-) \mathbf{X}\right) \\
&= \text{Tr}\left(\mathbf{X}^\top \mathbf{L} \mathbf{X}\right),
\end{aligned}$$

where matrix $\mathbf{L} = (\alpha \cdot \mathbf{L}^+ - (1-\alpha) \cdot \mathbf{L}^-)$ and α is the weight of the positive normalized cut cost term.

Moreover, the optimal community structure $\mathcal{C}^*$ (i.e., the optimal indicator matrix $\mathbf{X}^*$) which can minimize the *signed normalized cut* cost function can be represented as:

$$\begin{aligned}
&\mathbf{X}^* = \arg\min_{\mathbf{X}} \text{Tr}\left(\mathbf{X}^\top \mathbf{L} \mathbf{X}\right), \\
&s.t. \quad \mathbf{X}^\top \mathbf{X} = \mathbf{I}.
\end{aligned}$$

Constraint $\mathbf{X}^\top \mathbf{X} = \mathbf{I}$ ensures the obtained indicator matrix $\mathbf{X}$ is orthogonal. The discrete binary value constraint on $\mathbf{X}$ is usually relaxed, which can actually take any real values in range $[0, 1]$.

3.2 Objective Function of the SHED Problem

As introduced in Section 2, the *trust holes* either connecting different communities via positive links or linking individuals within communities via negative links will make the social community structure of the network hard to distinguish. Therefore, we propose to remove potential *trust hole* candidates together with their attached links from the network. The users removal of whom from the network can lead to the maximum community detection cost drop will be the optimal *trust holes* to be identified in the SHED problem. Based on the *signed normalized cut* cost function introduced in the previous section, we will define the concrete objective function of SHED in this section.

Let G, $\mathcal{C}^*$ and G', $(\mathcal{C}')^*$ be the networks and their optimal community structures before and after removing the *signed trust holes* $\mathcal{H}$ respectively. Considering that, in the *signed normalized cut* cost function, network information is actually stored in the Laplace matrix, next we will first study how to represent the Laplace matrix of network G' (i.e., $\mathbf{L}'$) after removing the *signed trust holes* from the original Laplace matrix of network G (i.e., $\mathbf{L}$).

Let matrix $\mathbf{I} \in \{0, 1\}^{|\mathcal{V}| \times |\mathcal{V}|}$ be the identity matrix with 1s on its diagonal only. Given the user set $\mathcal{V}$ and structure hole set $\mathcal{H} = \{u_i, u_j, \cdots, u_m\}$, we define the corresponding transformation matrix $\mathbf{T} \in \{0, 1\}^{(|\mathcal{V}| - |\mathcal{H}|) \times |\mathcal{V}|}$ based on $\mathbf{I}$, where the rows corresponding structure holes in $\mathcal{H}$ are all removed. For instance, if u_i is identified as a trust hole, after removing u_i, we can define the corresponding transformation to be $\mathbf{T} \in \{0, 1\}^{(|\mathcal{V}| - 1) \times |\mathcal{V}|}$, where rows $T(l, :) = I(l, :), \forall l \in \{1, 2, \cdots, i-1\}$ and $T(l, :) = I(l+1, :), \forall l \in \{i, i+1, \cdots, |\mathcal{V}| - 1\}$. Therefore, given a set of structure holes $\mathcal{H}$, we can define a unique transformation matrix $\mathbf{T}$ for it. In this paper, we will misuse matrix $\mathbf{T}$ to denote the signed trust hole for simplicity. With transformation matrix $\mathbf{T}$, we can represent the Laplace matrix to be $\mathbf{L}' = \text{Diag}(\mathbf{TAT}^\top) - \mathbf{TAT}^\top$, where $\mathbf{A}$ is the signed adjacency matrix of G weighted by parameter α.

Definition 4 (Signed NCut Decrease): Based on the Laplace matrices $\mathbf{L}$ and $\mathbf{L}'$ as well as transformation matrix $\mathbf{T}$, we can define the *signed ncut decrease* introduced by matrix $\mathbf{T}$ to be

$$\begin{aligned}&\text{NCut-Decrease}(\mathbf{T}) = \text{Ncut}(\mathcal{C}^*, G) - \text{Ncut}((\mathcal{C}')^*, G')\\&= \min_{\mathbf{X}} \text{Tr}\left(\mathbf{X}^\top \mathbf{L}\mathbf{X}\right) - \min_{\mathbf{X}'} \text{Tr}\left((\mathbf{X}')^\top \mathbf{L}'(\mathbf{X}')\right).\end{aligned}$$

Furthermore, the objective function for detecting the optimal *signed trust hole* $\mathcal{H}^*$ can be represented as

$$\begin{aligned}\mathcal{H}^* &= \arg\max_{\mathcal{H}} \text{NCut-Decrease}(\mathbf{T})\\&= \arg\max_{\mathbf{T}} \left(\min_{\mathbf{X}} \text{Tr}\left(\mathbf{X}^\top \mathbf{L}\mathbf{X}\right) - \min_{\mathbf{X}'} \text{Tr}\left((\mathbf{X}')^\top \mathbf{L}'(\mathbf{X}')\right) \right),\\s.t.\ &\mathbf{X}^\top\mathbf{X} = \mathbf{I}, (\mathbf{X}')^\top(\mathbf{X}') = \mathbf{I}, \mathbf{TT}^\top = \mathbf{I},\\&|\mathcal{H}| = h, \mathbf{T} \in \{0,1\}^{(|\mathcal{V}|-|\mathcal{H}|)\times|\mathcal{V}|}.\end{aligned}$$

Considering that matrix $\mathbf{T}$ is obtained from the identity matrix by removing rows corresponding to the structure hole users, the last constraint is added to ensure each row of $\mathbf{T}$ should contain only one entry with value 1, while the remaining entries are all 0s.

3.3 Solution to the SHED Problem

In this section, we will first analyze the objective function of the SHED problem first, and after that we will introduce an approximated method to address it.

Objective Function Analysis By studying the objective equation, we observe that the first constrained minimization equation is actually a constant, removal of which has no effects on the solutions. Therefore, we can simplify the objective function as follows

$$\begin{aligned}\mathcal{H}^* &= \arg\min_{\mathbf{T}} \min_{\mathbf{X}'} \text{Tr}\left((\mathbf{X}')^\top \mathbf{L}'(\mathbf{X}')\right),\\s.t.\ &(\mathbf{X}')^\top(\mathbf{X}') = \mathbf{I}, \mathbf{TT}^\top = \mathbf{I}, |\mathcal{H}| = h, \mathbf{T} \in \{0,1\}^{(|\mathcal{V}|-|\mathcal{H}|)\times|\mathcal{V}|}.\end{aligned}$$

As we can see, the objective function is actually a joint min-min non-linear integer programming problem involving multiple variables simultaneously, joint optimization of which is shown to be NP-hard [7]. Therefore, a new approximated method SCROLL is proposed in this paper to address the objective function based on an alternative updating schema. Constraint $|\mathcal{H}| = h$ will be removed from the objective function since the number of detected trust holes has been denoted by the dimension of matrix $\mathbf{T}$ already.

Solution SCROLL with Alternative Updating As introduced in the previous section, Laplace matrix $\mathbf{L}'$ can be represented as $\mathbf{L}' = \text{Diag}(\mathbf{TAT}^\top) - \mathbf{TAT}^\top$. The transformation matrix $\mathbf{T}$ is also involved in the $\text{Diag}(\cdot)$, which will make the partial derivatives calculation of the objective function about variable $\mathbf{T}$ infeasible. To address this problem, in this paper, we propose approximate the representation of $\mathbf{L}'$ as $\mathbf{L}' \approx \mathbf{TLT}^\top$ instead. The introduced deviation by such a approximation will be $\mathbf{TLT}^\top - (\text{Diag}(\mathbf{TAT}^\top) - \mathbf{TAT}^\top) = \mathbf{T} \cdot \text{Diag}(\mathbf{A}) \cdot \mathbf{T}^\top - \text{Diag}(\mathbf{TAT}^\top)$, which are mainly about the values (about the out-degrees of the trust holes) on the diagonal of $\mathbf{L}'$. Based on such an approximation, the new objective function can be represented as

$$\begin{aligned} &\mathcal{H}^* = \arg\min_{\mathbf{T}} \min_{\mathbf{X}'} \text{Tr}\left((\mathbf{X}')^\top \mathbf{TLT}^\top (\mathbf{X}')\right), \\ &s.t.\ (\mathbf{X}')^\top (\mathbf{X}') = \mathbf{I}, \mathbf{TT}^\top = \mathbf{I}. \end{aligned}$$

Here the integer constraint matrix $\mathbf{T}$ is relaxed and entries in $\mathbf{T}$ can take any real values in range $[0, 1]$.

We propose to address the objective function with an alternative updating schema: (1) fix $\mathbf{T}$ and update $\mathbf{X}'$; and (2) fix $\mathbf{X}'$ and update $\mathbf{T}$.

Step 1: By fixing variable $\mathbf{T}$ and adding the constraint term $(\mathbf{X}')^\top (\mathbf{X}') = \mathbf{I}$ as a regularizer term, we can represent the objective function to be

$$\min_{\mathbf{X}'} \text{Tr}\left((\mathbf{X}')^\top \mathbf{TLT}^\top (\mathbf{X}')\right) + \rho \left\|(\mathbf{X}')^\top (\mathbf{X}') - \mathbf{I}\right\|_F^2,$$

where parameter ρ denotes the weight of the regularizer term, and it is assigned with with very large value (e.g., 10) in the experiment to ensure the constraint can be maintained.

The above objective function is a convex function can be addressed with gradient descent method, and the updating equation of variable $\mathbf{X}'$ can be represented as

$$\begin{aligned} \mathbf{X}' &= \mathbf{X}' - \eta_1 \frac{\partial \Big(\text{Tr}\left((\mathbf{X}')^\top \mathbf{TLT}^\top \mathbf{X}'\right) + \rho \left\|(\mathbf{X}')^\top \mathbf{X}' - \mathbf{I}\right\|_F^2 \Big)}{\partial \mathbf{X}'} \\ &= \mathbf{X}' - 2\eta_1 \Big(\mathbf{TLT}^\top \mathbf{X}' + 2\rho\big(\mathbf{X}'(\mathbf{X}')^\top \mathbf{X}' - \mathbf{X}'\big)\Big), \end{aligned}$$

where parameter η_1 denotes the learning step and it is assigned with a small constant value (0.0001) in the experiments.

Step 2: Meanwhile, in a similar way, by fixing parameter $\mathbf{X}'$ and adding the constraint term $\mathbf{TT}^\top = \mathbf{I}$ as a regularizer term, we can find that the resulting objective function is also a convex function. We can further represent the updating equation of variable $\mathbf{T}$ to be

$$\mathbf{T} = \mathbf{T} - 2\eta_2 \big(\mathbf{X}'(\mathbf{X}')^\top \mathbf{TL} + 2\rho(\mathbf{T}(\mathbf{T})^\top \mathbf{T} - \mathbf{T})\big),$$

where parameter η_2 denotes the learning step of updating $\mathbf{T}$.

Table 1. Properties of different networks

network	# nodes	# links	link type
Epinions	131,828	841,372	directed
Slashdot	77,350	516,575	directed

Therefore, the alternative updating equation of variables $\mathbf{X}'$ and $\mathbf{T}$ at step τ can be represented as

$$\begin{cases} \mathbf{X}'^{(\tau)} & = (\mathbf{X}')^{(\tau-1)} - 2\eta_1 \Big(\mathbf{T}^{(\tau-1)} \mathbf{L} (\mathbf{T}^{(\tau-1)})^\top (\mathbf{X}')^{(\tau-1)} \\ & +2\rho\big((\mathbf{X}')^{(\tau-1)}((\mathbf{X}')^{(\tau-1)})^\top(\mathbf{X}')^{(\tau-1)} - (\mathbf{X}')^{(\tau-1)}\big)\Big), \\ \mathbf{T}^{(\tau)} & = \mathbf{T}^{(\tau-1)} - 2\eta_2\big((\mathbf{X}')^{(\tau)}((\mathbf{X}')^{(\tau)})^\top(\mathbf{T}')^{(\tau-1)}\mathbf{L} \\ & +2\rho(\mathbf{T}^{(\tau-1)}(\mathbf{T}^{(\tau-1)})^\top\mathbf{T}^{(\tau-1)} - \mathbf{T}^{(\tau-1)})\big). \end{cases}$$

Such a alternative updating process will continue until both $\mathbf{X}'$ and $\mathbf{T}$ converge. From the result of $\mathbf{T}$, we can recover the rows that are removed from the identified results, which corresponding to the *signed trust holes* of the signed network. Under the constraint that each row and each column can constrain at most one entry being filled with value 1, for entries in $\mathbf{T}^{(\tau)}$, we sort their values in decreasing order to select the entries with the largest values to preserve, which will be assigned with 1. The rest will all be assigned with value 0. In addition, based on matrix $\mathbf{X}'$, we can obtain the community structures formed by users in the signed networks. Depending on the positions and the connections attached to the identified *trust holes* (i.e., intra or inter community, and positive or negative links), we can differentiate the specific categories of *trust holes* (i.e., *positive* and *negative trust holes* respectively) from the results.

4 Experiments

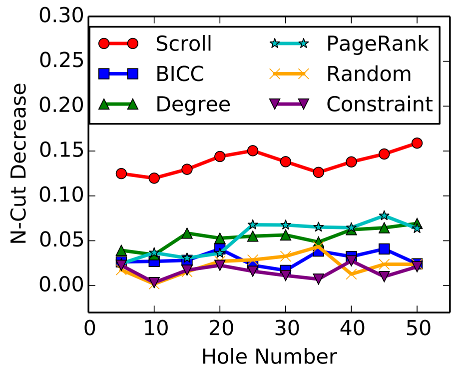
To test the effectiveness of SCROLL in addressing the SHED problem. We conduct extensive experiments on real-world signed network datasets, and compare them with both state-of-art and traditional baseline methods.

4.1 Dataset Description

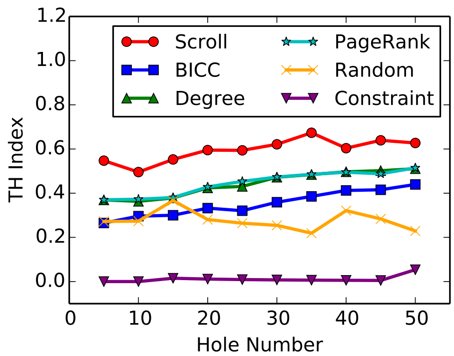
The real-world signed network dataset used in the experiments include the Epinions network and the Slashdot network. Some basic statistical information about these two datasets is available in Table 1.

Reproducible Research?: The dataset used in this paper is public accessible, which can be downloaded from the SNAP site[2].

[2] http://snap.stanford.edu/data/index.html#signnets



(a) N-Cut Decrease



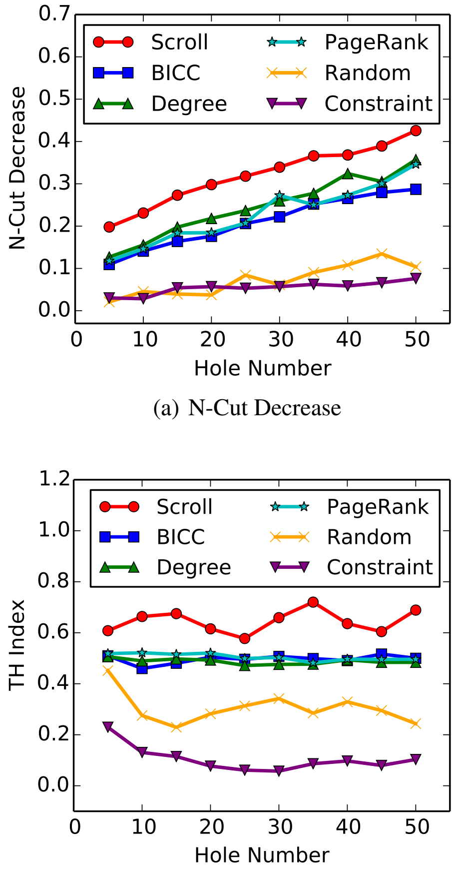
(b) TH Index

Fig. 2. Experiment results on the Epinions network.

4.2 Experiment Setting

Based on the positive and negative links, the positive and negative adjacency matrices are constructed respectively. With the positive/negative adjacency matrices, we can define the integrated Laplace matrix and the weight of positive Laplace matrix α is set as 0.5. Framework SCROLL infers the transformation matrix $\mathbf{T}$ and confidence matrix $\mathbf{X}$ simultaneously with the alternative updating schema. From the transformation matrix $\mathbf{T}$ we can recover the trust holes.

Comparison Methods The networks studied in this paper are signed, and no existing works have studied the trust hole detection problem based on signed networks before yet. We propose to apply the existing works on traditional structural hole detection problem to the signed networks by discarding the polarity information. In SHED, no community structure information is available about the signed networks, thus the unsigned structural hole detection method proposed in [15] taking the community structure as the



(a) N-Cut Decrease

(b) TH Index

Fig. 3. Experiment results on the Slashdot network.

input cannot work for the SHED problem. The list of comparison methods used in the experiments are provided as follows:

- SCROLL: Method SCROLL is the trust hold detection method proposed in this paper, which can consider both the links as well as the polarities attached to the links.
- BICC: Method BICC is the state-of-art structural hole detection method for unsigned networks [18]. To accommodate the setting of BICC, we transform the signed networks to a unsigned one by discarding the link polarities.
- CONSTRAINT: Method CONSTRAINT proposed in [2] uses constraint to estimate the importance of each node and selects the nodes with the lowest K constraint scores as the hole candidates.

Table 2. Intersection of trust holes selected by different comparison methods.

Epinions

	SCROLL	BICC	CONSTRAINT	PAGERANK	DEGREE	RANDOM
SCROLL	50	26	1	32	25	4
BICC		50	2	25	26	3
CONSTRAINT			50	1	1	2
PAGERANK				50	22	4
DEGREE					50	4
RANDOM						50

Slashdot

	RANDOM	DEGREE	PAGERANK	CONSTRAINT	BICC	SCROLL
SCROLL	4	35	27	4	23	50
BICC	5	23	23	4	50	
CONSTRAINT	0	4	5	50		
PAGERANK	6	27	50			
DEGREE	4	50				
RANDOM	50					

– PAGERANK: Traditional node ranking algorithm PAGERANK can also be used as a method for structural hole detection in [15], which returns the nodes with the top K page rank scores as the result.
– DEGREE: Method DEGREE selects the users with the top K degree as the hole candidates. Considering that the links in the network datasets are directed, the social degree of user u is identical to the number of neighbors of u in the network (i.e., $|\Gamma(u)| = |\{v|v \in \mathcal{U}, (u, v) \in \mathcal{E} \vee (v, u) \in \mathcal{E}\}|$).
– RANDOM: Method RANDOM randomly selects K users as the hole candidates.

Evaluation Metrics For the real-world signed network datasets, we propose to measure the performance of these different comparison methods two different metrics. One of the evaluation metrics is the *signed normalized cut decrease* introduced in this paper. Generally, higher *signed normalized cut decrease* corresponds to better performance.

Another evaluation metric used in the experiments is the *trust hole index* introduced in this paper. Generally, the optimal trust holes are the user nodes which connect to other nodes in different communities via the positive links or nodes in the same community via negative links.

Definition 5 (Trust Hole Index): Let $\Gamma^-(u)$ and $\Gamma^+(u)$ be the sets of negative and positive neighbors of user u respectively and mapping $c : \mathcal{U} \to \mathcal{C}$ be the function projects users to their communities respectively. The *trust hole index* for user u can be represented as

$$\begin{aligned}\text{TH-Index}(u) = &\frac{1}{Z(\Gamma^+(u))} \sum_{v,w \in \Gamma^+(u), v \neq w} I(c(v) \neq c(w)) \\ &+ \frac{1}{Z(\Gamma^-(u))} \sum_{v,w \in \Gamma^-(u), v \neq w} I(c(v) = c(w)).\end{aligned}$$

where the normalization term $Z(\Gamma^+(u)) = \frac{1}{|\Gamma^+(u)| \times (|\Gamma^+(u)|-1)}$, and term $Z(\Gamma^-(u)) = \frac{1}{|\Gamma^-(u)| \times (|\Gamma^-(u)|-1)}$. Function $I(c(v) \neq c(w))$ is 1 if $c(v) \neq c(w)$ and similar for $I(c(v) = c(w))$.

Generally, users with higher *trust hole index* are more likely to be the trust holes, and methods achieving higher *trust hole index* will have better performance.

4.3 Experiment Result

The experiment results on the real-world signed networks, Epinions and Slashdot, are shown in Figure 2 and Figure 3 respectively.

In Figure 2(a), we show the performance of different comparison methods evaluated by the *normalized cut decrease* metric at different hole numbers. From the plot, we can observe that SCROLL can outperform other comparison methods for different hole numbers. For instance, when the hole number is 25, the *normalized cut decrease* achieved by SCROLL is 0.150, which is more than two times higher than that obtained by DEGREE and PAGERANK, about 7 times larger than that achieved by BICC, RANDOM and CONSTRAINT. In Figure 2(b), we show the performance of the comparison methods when the evaluation metric is *trust hole index*. According to the plot, we can observe that DEGREE and PAGERANK has comparable performance, both of which are below the *trust hole index* curve of SCROLL. The constraint method cannot work well when dealing with the signed networks at all, which is the baseline of all the comparison methods.

Similar results of these comparison methods can be observed for the Slashdot network in Figure 3, and SCROLL can outperform other methods for different hole numbers.

In addition, in Table 2, we show the shared trust holes detected by the different comparison methods in the Epinions and Slashdot networks respectively. From the results, we can observe that the trust holes selected by SCROLL have some overlapping with BICC, DEGREE and PAGERANK. In the Epinions, the number of detected trust holes by SCROLL and BICC is 26, the number of shared trust holes by SCROLL and PAGERANK is 32, while those shared by SCROLL and DEGREE is 25 respectively. Meanwhile, the trust holes detected by these three methods are quite different from those selected by RANDOM and CONSTRAINT. For instance, in Epinions, the trust holes shared by SCROLL and CONSTRAINT is merely 1, and those shared by SCROLL and RANDOM is only 4. Similarly results can be observed for the Slashdot network, and SCROLL can choose some common holes with BICC, PAGERANK and DEGREE, but have quite different results with RANDOM and CONSTRAINT.

5 Related Works

Traditional structural holes are usually correlated to a wide range of indicators about social success, which have been studied in various papers already [1, 3, 4]. Ahuja [1] proposes to study the effects of a firm's network of relations on innovation from three different perspectives: direct ties, indirect ties and structural holes, where structural holes are discovered to have both positive and negative influences on subsequent innovations. Burt [3] introduces the relation between structural holes and good ideas. Burt discovers that structural holes connecting different groups are more likely to express ideas, less likely to have ideas dismissed, and more likely to have ideas evaluated as valuable.

Later, some works propose to study the formation of structural holes in social networks from the game theory perspective [5, 8, 9]. Goyal et al. [8] propose that in social networks, individuals form links with others to create surplus, to gain intermediation rents, and to circumvent others, which are the forces in the process of strategic network formation. Kleinberg et al. [9] propose to apply a game-theoretic approach to study the structural holes, and notice that individuals will differentiate themselves in equilibrium of the game, occupying different social strata and receiving different payoffs.

Recently, some works have been done on finding the structure holes from social networks [15, 23]. Lou et al. [15] formulate the structure hold mining problem from the information diffusion and community detection perspectives, and discover that the problem is NP-hard based on these two modeling. Vilhena et al. [23] extend the structural hole concept to "culture holes" and propose to find the "culture holes" from the citation networks. For more background knowledge about online social networks and the research works studied based on them, please refer to a recent survey paper written by Shi et al. in [19].

Signed networks since introduced by Leskovec et al. [12] have become a hot research topic, as links in signed networks can denote different attitudes among users, which provide new opportunities for researchers to study the connections among users. Leskovec et al. [11] propose to predict the positive and negative links in signed networks based on the balance theory. Doreian et al. [6] study the partition problem in signed networks. A recent survey about related works in signed networks is given by Tang et al. in [21].

6 Conclusion

In this paper, we have studied the *trust hole* detection problem in signed networks. A formal definition about the *trust hole* concept as well as its two different variants are clearly illustrated in this paper. To identify the set of trust holes from the signed network, a community detection based trust hole detection framework, SCROLL, is introduced in this paper. By identifying the set of users, removal of whole from the network can lead to the maximum *signed normalized cut decrease*, SCROLL can detect the optimal set of trust holes from the signed network. Extensive experiments have been done on real-world signed networks, and the results demonstrate the effectiveness of SCROLL in addressing the SHED problem.

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